

Explicit computations of Siegel modular forms of degree two.

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ABSTRACT

Unlike classical modular forms, there is currently no general way to implement the computation of Siegel modular forms of arbitrary weight, level and character, even in degree two. There is however, a way to do it in a unified way. After providing a survey of known computations we describe the implementation of a class modeling Siegel modular forms of degree two in Sage. In particular, we describe algorithms to compute a variety of rings of Siegel modular forms, many of which are implemented in our class. A wide variety of Siegel modular forms (e.g., both vector- and scalar-valued) can be modeled via this class and we unify these via a construct we call a formal Siegel modular form. We define this notion and discuss it in detail.

1. Introduction

Computing modular forms is of broad interest. For example, the explicit computation of elliptic modular forms has led to a number of conjectures that in many cases have become theorems. Similar, but neither as comprehensive nor as systematic, computations have been carried out with Siegel modular forms as well. A consideration of Siegel modular forms and their Fourier expansions aiming at an implementation should suit two purposes. Firstly, one has to obtain the generators of a space or ring of modular forms as easily and quickly as possible. Secondly, one wants to construct more complicated modular forms and recognize them as a linear or algebraic combination of the generators.

Computing elliptic modular forms of level 1 is rather straightforward. It is well-known that the ring of such forms is a polynomial ring generated by the Eisenstein series E_4 and E_6 . Alternatively, the weight k space is spanned by E_k and all the products $E_i E_{k-i}$ for i running from 4 to $k-2$ as can be proved by means of methods found in [31]. Computations in level bigger than one are typically done via the theory of modular symbols. For instance, this is how both MAGMA [6] and Sage [52] compute spaces of elliptic modular forms.

An elliptic modular form can be thought of as a Siegel modular form of degree 1. The theory of the computation of Siegel modular forms of even degree 2 is not as general as that for degree 1. There is currently no theory of modular symbols. What there is, though, is an explicit description of a number of particular rings of Siegel modular forms. More generally, by a theorem of Igusa, we know that every ring associated to any finite index subgroup of the full Siegel modular group is generated by polynomials in theta constants (cf. [29, 30]). This theorem has been exploited by Igusa [27, 28] to explicitly list the 5 generators of the ring of Siegel modular forms of degree 2. A more recent application of this theorem in a number of papers [2, 21, 23] by Ibukiyama and his collaborators have characterized rings of Siegel modular forms for levels up to 4 as polynomial rings in explicitly described generators. These generators are typically described in terms of theta constants.

Another method to compute spaces of Siegel modular forms (and thereby, possibly rings of modular forms) is Poor and Yuen’s restriction technique (see for example, [43]). In their

papers on this topic they are typically trying to compute forms in level 1 and degree larger than 2, but in principle their technique should work in degree 2. The potential obstacle when computing for level bigger than one is that there are several cusps and one needs to know the expansion at each of them. The method proceeds roughly as follows: they define a linear map from spaces of degree n Siegel modular forms down to degree 1. They search for linear relations among the Fourier coefficients of degree 1 forms and pull them back to linear relations among the Fourier coefficients of degree n forms. This allows them to compute the dimensions of particular spaces of degree n Siegel modular forms and to write down bases for those spaces. To determine these bases, they compute theta series, Ikeda lifts, Eisenstein series, and theta constants (see Section 3).

The above two families of results are both for scalar-valued Siegel modular forms. In some cases (e.g., [48]), one can explicitly compute spaces of vector-valued Siegel modular forms. An example of such a space is given in terms of the 4 Igusa generators of even weight and an explicitly described differential operator (see Section 3.8.1). We limit our attention to spaces of vector-valued Siegel modular forms because the ring of such forms is not finitely generated.

In addition to providing a useful survey of what is known about the computation of degree 2 Siegel modular forms (e.g., what rings can be computed, what vector spaces, etc.), we describe an implementation [46] of the types of Siegel modular forms often used to describe generators. We provide algorithms that are used in our implementation in Sage [52] and that are improvements (sometimes minor) over the naive algorithms that follow immediately from theorems in the literature. Another feature of our implementation is that it allows for algebraic combinations of vector- and scalar-valued Siegel modular forms. In fact, our class was designed in such a general way that one could use the same basic class to implement a wide variety of automorphic forms.

Often one is interested in computing Hecke eigenvalues associated to modular forms as opposed to computing Fourier coefficients. In our work we are mainly interested in computing Fourier coefficients but now mention recent work that allows for the computation of Hecke eigenvalues. It is possible to compute Hecke eigenvalues by a purely cohomological approach. Until now this has been particularly useful in the case of vector-valued Siegel modular forms [15, 16]. The trace of the Hecke action on the full cohomology can be determined by counting curves modulo p . These calculations are demanding in terms of theory and raw computational power.

The paper is organized as follows. In Section 2 we give the definition of Siegel modular forms and provide the definition of formal Siegel modular forms: this is what our Sage package actually implements. In Section 3, we do two things when appropriate. First, for a wide variety of types of Siegel modular forms of degree 2, we give a survey of known theorems related to their computation. Second, we give a description of the algorithm we use to compute Siegel modular forms of that type and any theorems related to the algorithm. Then, in Section 4 we list the spaces and rings that are described in the literature and that we have computed. The final two sections deal with the implementation, its details and the choices we have made. We also provide links to data for modular forms in those rings for which we have computed data. In Section 5 we describe in broad strokes the class we designed, paying particular attention to how we multiply Siegel modular forms of different types and to how we model the precision of a given form. In Section 6 we briefly point out how other spaces of automorphic forms can be modeled via a class similar to the one we describe.

2. Siegel Modular Forms

In this section, we define Siegel modular forms of degree 2. If one were to implement this definition directly, one would have a rather inefficient class as it would be essentially a dense

power series ring in three variables. What we end up implementing in Sage is a *formal* Siegel modular, a notion we define in this section. Also in this section we show that formal Siegel modular forms do model the Fourier expansions of Siegel modular forms rather well. A formal Siegel modular form is in some sense more general than a Siegel modular form: the set of Fourier expansions of Siegel modular forms is a subset of the Fourier expansions modeled by formal Siegel modular forms. We sometimes take advantage of this fact in the implementations of our algorithms. At the same time, however, it respects as much structure as needed to store the data associated to the modular form in a compact way.

2.1. Basic Definitions

We recall the definition of a Siegel modular form of degree two. We will consider the symplectic group $\mathrm{Sp}(2, \mathbb{Q})$ of degree 2 over \mathbb{Q} ; i.e., the group of matrices $M \in \mathrm{M}(4, \mathbb{Q})$ such that ${}^t M J M = J$. Here $J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$ represents the standard symplectic form. The full Siegel modular group $\mathrm{Sp}(2, \mathbb{Z}) = \mathrm{Sp}(2, \mathbb{Q}) \cap \mathrm{M}(4, \mathbb{Z})$ will be the central object of study. For any $\Gamma \subset \mathrm{Sp}(2, \mathbb{Q})$ commensurable to $\mathrm{Sp}(2, \mathbb{Z})$, let $\psi : \Gamma \rightarrow \mathbb{S}^1$ be a character and k, j be nonnegative integers. Let

$$\mathbb{H}_2 := \{Z = X + iY \in \mathrm{M}(2, \mathbb{C}) : {}^t Z = Z, Y > 0\}$$

be the Siegel upper half space of degree 2. Assign to $\mathbb{C}[X, Y]_j$, the space of homogeneous polynomials of degree j , the $\mathrm{GL}(2, \mathbb{C})$ -action

$$(A, p) \mapsto A \cdot p := p((X, Y)A). \quad (2.1)$$

Now we can define Siegel modular forms:

DEFINITION 1. A Siegel modular form of degree 2, weight (k, j) , and character ψ on Γ is a complex analytic function $F : \mathbb{H}_2 \rightarrow \mathbb{C}[X, Y]_j$ such that

$$F(gZ) := F((AZ + B)(CZ + D)^{-1}) = \psi(g) \det(CZ + D)^k (CZ + D) \cdot F(Z)$$

for all $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$.

The space of all such functions is denoted $M_{k,j}(\Gamma, \psi)$, where we suppress ψ if it is the trivial character and j if it is 0. If j is positive F is called vector-valued, otherwise it is called scalar-valued. We use $M_*(\Gamma) := \bigoplus_k M_k(\Gamma)$ for the ring of (scalar-valued) Siegel modular forms of degree 2 on Γ .

Common choices of Γ and ψ include $\Gamma = \Gamma_0(\ell) := \left(\begin{smallmatrix} \mathbb{Z}^{2 \times 2} & \mathbb{Z}^{2 \times 2} \\ \ell \mathbb{Z}^{2 \times 2} & \mathbb{Z}^{2 \times 2} \end{smallmatrix} \right) \cap \mathrm{Sp}(2, \mathbb{Z})$, and $\psi \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \tilde{\psi}(\det D)$ where $\tilde{\psi}$ is a Dirichlet character mod ℓ . A Siegel modular form on $\Gamma_0(\ell)$ and with such a character ψ is commonly referred to as a Siegel modular form of level ℓ and character $\tilde{\psi}$.

2.2. Fourier Coefficients and Formal Siegel Modular Forms

Having defined analytic Siegel modular forms of degree 2, we now define a formal Siegel modular form. Every analytic Siegel modular form F can be identified by its Fourier expansion. Because of a certain invariance (see Proposition 2.1 for a precise statement) that the coefficients of F possess, the form F can be identified by its Fourier coefficients indexed on $\mathrm{GL}(2, \mathbb{Z})$ -reduced quadratic forms. So, we define a formal Siegel modular form C_F related to F as a function from integral binary quadratic forms to the representation space of F .

We denote by Q the set of all integral binary quadratic forms that are positive semidefinite:

$$Q := \{f = [a, b, c] : a, b, c \in \mathbb{Z}, b^2 - 4ac \leq 0, a \geq 0\}.$$

For $f = [a, b, c]$ we use $M_f = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$. The group $\mathrm{GL}(2, \mathbb{Z})$ acts on Q by $M_{A \cdot f} = A M_f {}^t A$.

Let F be a Siegel modular form on Γ . It is not a serious restriction to assume, as we shall do for the following, that $P := \begin{pmatrix} I_2 & \mathbb{Z}^{2 \times 2} \\ 0_2 & I_2 \end{pmatrix} \cap \mathrm{Sp}(2, \mathbb{Z}) \subset \Gamma$, since this can always be achieved by passing from Γ to a $\mathrm{Sp}(2, \mathbb{Q})$ -conjugate subgroup. Moreover, we also assume that ψ is trivial on P . The invariance of F under the action of P implies that F is periodic and therefore has a Fourier expansion of the form

$$\begin{aligned} F(Z) &= \sum_{f=[a,b,c] \in Q} C_F(f) e(a\tau + bz + c\tau') \\ &= \sum_{f=[a,b,c] \in Q} C_F(f) e(\mathrm{tr}(ZM_f)) \end{aligned}$$

Here $Z := \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ ($\tau, \tau' \in \mathbb{H}_1$ and $z \in \mathbb{C}$), $e(x) = e^{2\pi i x}$, and the trace of a matrix A is denoted by $\mathrm{tr} A$.

Let $A^* := {}^t(A^{-1})$. Assume

$$\begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix} \in \Gamma \text{ for some } A \in \mathrm{GL}(2, \mathbb{Z}). \quad (2.2)$$

Then,

$$F\left(\begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix} Z\right) = F(A^* Z A^{-1}) = \psi\left(\begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix}\right) \det(A)^k A \cdot F(Z).$$

In other words, equating the Fourier expansion of the left-hand side to that of the right-hand side, we get

$$\sum_f C_F(f) e(\mathrm{tr}(A^* Z A^{-1} M_f)) = \psi\left(\begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix}\right) \det(A)^k \sum_f A \cdot C_F(f) e(\mathrm{tr}(Z M_f)). \quad (2.3)$$

Using the fact that the trace is invariant under cyclic permutations of the matrices being multiplied, we deduce

$$\mathrm{tr}(A^* Z A^{-1} M_f) = \mathrm{tr}(Z A^{-1} M_f A^*) = \mathrm{tr}(Z M_{A^{-1} \cdot f})$$

where the last equality follows from the definition of the $\mathrm{GL}(2, \mathbb{Z})$ action on Q . Replacing f by $A \cdot f$ in the left-hand side of (2.3) yields the following:

PROPOSITION 2.1. *Let the notation be as above. Then*

$$C_F(A \cdot f) = \psi\left(\begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix}\right) \det(A)^k A \cdot C_F(f). \quad (2.4)$$

for all A in $\mathrm{GL}(2, \mathbb{Z})$, such that $\begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix} \in \Gamma$.

This allows us to define a formal Siegel modular form.

DEFINITION 2. Let R be a module (or ring) with a left $\mathrm{GL}(2, \mathbb{Z})$ action, and let χ be a linear character of $\mathrm{GL}(2, \mathbb{Z})$. A formal Siegel modular form with values in R and (formal) character χ is a map $C : Q \rightarrow R$ for which

$$C(A \cdot f) = \chi(A) A \cdot C(f) \quad (2.5)$$

for all $A \in \mathrm{GL}(2, \mathbb{Z})$.

By Proposition 2.1, we see that this definition models a Siegel modular form well, with the formal character given by

$$\chi(A) = \psi\left(\begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix}\right) \det(A)^k.$$

Moreover, it is enough to store the values of the coefficient map C at the $\mathrm{GL}(2, \mathbb{Z})$ -reduced elements of Q , since the other coefficients can be computed using (2.5).

We note that there are only 4 possibilities for the character of a formal Siegel modular form:

$$\chi \in \{1, \det, \sigma, \det\sigma\}$$

where $\sigma(A)$ is the sign of $A \bmod 2$ acting as a permutation on the set $\{(0, 1), (1, 0), (1, 1)\}$. In fact, the group of linear characters of $\mathrm{GL}(2, \mathbb{Z})$ is non-cyclic of order 4 generated by \det and σ . This follows from the fact that the abelianization of $\mathrm{GL}(2, \mathbb{Z})$ is isomorphic to the Klein four group. Indeed, $\mathrm{GL}(2, \mathbb{Z})$ is generated by the matrices $E = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. From $(ST)^3 = S^2$ and $ETE = T^{-1}$ we deduce $\pi(S) = \pi(T)^{-3}$ and $\pi(T) = \pi(T)^{-1}$, where π is the canonical map onto the abelianization of $\mathrm{GL}(2, \mathbb{Z})$. Therefore, the abelianization is generated by $\pi(E)$ and $\pi(T)$, both of which have order 1 or 2. On the other hand, since we have four characters, the order of the abelianization is at least four. This implies our claim.

For a Siegel modular form of level ℓ , the formal character can only be either 1 or \det , depending only on the parity of the weight k and the parity of the character ψ .

2.3. Formal Constructions

In order to highlight the usefulness of thinking of Siegel modular forms as purely formal objects, we now describe two analytic constructions from this formal point of view.

2.3.1. Siegel Φ operator We now describe the Siegel Φ operator formally. The map Φ takes a degree n Siegel modular form to a degree $n - 1$ Siegel modular form. In particular, cusp forms are in the kernel of Φ and Eisenstein series are mapped to Eisenstein series. The Siegel Φ operator for degree 2 Siegel modular forms has a very simple formula in terms of its singular Fourier coefficients:

$$\Phi(F) := \sum_{n \geq 0} C_F([0, 0, n]) q^n. \quad (2.6)$$

2.3.2. Hecke Operators In this section we give formulas for the image of a Siegel modular form of level N and weight (k, j) . We give two formulas. The first is for the operator $T(p^\delta)$ when p and N are coprime. The second is for the operator $U(p)$ when p divides the level.

Let F be a Siegel modular form as above. Let C be its corresponding formal Siegel modular form and let C' be the formal Siegel modular form that corresponds to the image of F under $T(p^\delta)$. Then, as found in [26],

$$C'([a, b, c]) = \sum_{\alpha+\beta+\gamma=\delta} p^{\beta(k+j-2)+\gamma(2k+j-3)} \sum_{\substack{U \in R(p^\beta) \\ a_U \equiv 0 \pmod{p^{\beta+\gamma}} \\ b_U \equiv c_U \pmod{p^\gamma}}} (d_{0,\beta} U) \cdot C \left(p^\alpha \left[\frac{a_U}{p^{\beta+\gamma}}, \frac{b_U}{p^\gamma}, \frac{c_U}{p^{\gamma-\beta}} \right] \right)$$

where $R(p^\beta)$ is a complete set of representatives for $\mathrm{SL}(2, \mathbb{Z})/\Gamma_0^{(1)}(p^\beta)$; where, for $f = [a, b, c]$, $[a_U, b_U, c_U] = f_U := f((X, Y)^t U)$; where $d_{0,\beta} = \begin{pmatrix} 1 & \\ & p^\beta \end{pmatrix}$; and where the dot denotes the action of $d_{0,\beta} U$ as defined in (2.1). We note that if F is scalar-valued this action is trivial.

For primes p dividing the level of a scalar-valued Siegel modular form F , there is an operator $U(p)$. In [4], Böcherer gives an easy explicit formula for the action of $U(p)$ on a Siegel modular form. In particular, let C and C' be formal Siegel modular forms corresponding to F and the image of F under $U(p)$. Then

$$C'([a, b, c]) = C([pa, pb, pc]).$$

3. Recipes and Algorithms

The generators of rings of Siegel modular forms of degree 2 are often given in one of the following ways: a linear combination of theta series, a linear combination of products of theta constants, or, if the form is of level 1, it may be given in terms of the Igusa generators. Additionally, particular Siegel modular forms of degree 2 can also be written in terms of Maass lifts, Eisenstein series, or Borcherds products. In this section, we present the definitions of all these terms and algorithms or recipes that can be used to compute them (or, at the very least, provide references to such methods).

3.1. Eisenstein Series

We define an Eisenstein series of degree 2 by

$$E_k(Z) = \sum_{\{C,D\}} \det(CZ + D)^{-k}$$

where the sum is over nonassociated symmetric coprime matrices $C, D \in M_2(\mathbb{Z})$ (here, two integral matrices A and B are called coprime if the matrix products GA and GB are integral when and only when G is integral). Earlier work by Maass [32] and Mizumoto [35, 36] give explicit formulas for the Fourier coefficients of Eisenstein series of level 1. We describe another approach here.

In [14] one can find a formula for the Fourier coefficients of a Siegel Eisenstein series of degree 2. The formula is in terms of Cohen's function which we define now. Let

$$L_D(s) = \begin{cases} 0 & \text{if } D \not\equiv 0, 1 \pmod{4} \\ \zeta(2s-1) & \text{if } D = 0 \\ L(s, \left(\frac{D_0}{\cdot}\right) \sum_{d|f} \mu(d) \left(\frac{D_0}{d}\right) d^{-s} \sigma_{1-2s}(f/d)) & \text{if } D \equiv 0, 1 \pmod{4} \text{ and } D \neq 0 \end{cases}$$

where D has been written as $D = D_0 f^2$ with $f \in \mathbb{N}$, D_0 is the discriminant of $\mathbb{Q}(\sqrt{D})$ and $\sigma_k(n) = \sum_{d|n} d^k$. We define Cohen's function by $H(k-1, N) = L_{-N}(2-k)$.

Now suppose C_k is the formal Siegel modular form corresponding to the Eisenstein series E_k . Then, according to [14, Cor. 2, p. 80], we have

$$C_k([a, b, c]) = \sum_{d|(a,b,c)} d^{k-1} H\left(k-1, \frac{4ac-b^2}{d^2}\right).$$

We observe that the coefficient of a singular form $C_k([0, 0, c])$ is $\zeta(3-2k)\sigma_k(c)$, which is the c th coefficient of the Eisenstein series of degree 1 and weight $2k-2$. We note that one can compute Cohen's function within Sage.

We remark that the weight k Eisenstein series can also be computed as the Saito-Kurokawa lift of the degree 1, weight $2k-2$ Eisenstein series. In fact, that's where the formulas in this section come from. See Section 3.3 for more details on how to compute Saito-Kurokawa lifts.

One can also consider Eisenstein series with level. Mizuno [37, Theorem 1] gives a very complicated formula for Eisenstein series of degree 2 and odd squarefree level. The formula consists of special values of L-functions, Gauss sums, and the like. We refer the interested reader to that paper for details.

3.2. Klingen-Eisenstein Series

Given a modular form f of degree $n_1 \leq n$ and weight $k > n + n_1 + 1$ the Klingen-Eisenstein series of f is a modular form $E_{n,n_1}(f)$ of degree n and weight k such that the Siegel Φ operator

maps $E_{n,n_1}(f)$ to f . More precisely, it is given by the summation

$$E_{n,n_1}(f) = \sum_{M \in \Gamma_{n_1,\infty}^{(n)} \setminus \Gamma^{(n)}} f|M, \quad (3.1)$$

where f is considered a function on $\mathbb{H}_n \leftarrow \mathbb{H}_{n_1}$ and $\Gamma_{n_1,\infty}^{(n)}$ is the n_1 -parabolic subgroup

$$\Gamma_{n_1,\infty}^{(n)} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} : C_{ij} = 0 \text{ for } i > n_1, \text{ all } j, D_{ij} = 0 \text{ for } i > n_1, j \leq n_1 \right\}, \quad (3.2)$$

where the ij th entry of a matrix M is denoted M_{ij} .

We restrict to $n = 2$. Then $E_{2,0}(f) = E_2$ and $E_{2,2}(f) = f$ as can be seen from the easy way in which the corresponding n_1 -parabolic subgroups can be described. So we have to give formulas for $E_{2,1}$ and this was done by Mizumoto in [35, 36]. We can restrict to the case that f is an eigenform. Then, we can assume that f is cuspidal, since otherwise $E_{2,1}(f)$ is a classical Eisenstein series.

Mizumoto gives an explicit formula for the primitive coefficients of a Klingen-Eisenstein series in terms of the coefficients of f and special values of three different L-functions. We do not present this explicit formula, but note, instead, that one can easily identify the Klingen Eisenstein series among a basis of eigenforms. And this is how we compute their Fourier expansions in practice.

3.3. Maass Lifts

This lift can be thought of as a modular form that can, in some sense, be computed easily from modular forms that are easier to compute. In particular, it can be derived from Jacobi forms and elliptic modular forms. The standard reference for Jacobi forms is [14].

DEFINITION 3. A Jacobi form of level 1, weight k and index m is a function $\phi(z, \tau)$ for $z \in \mathbb{H}_1$ and $\tau \in \mathbb{C}$ such that:

- (i) $\phi\left(\frac{az+b}{cz+d}, \frac{\tau}{cz+d}\right) = (cz+d)^k e^{\frac{2\pi i m c \tau^2}{cz+d}} \phi(\tau, z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$;
- (ii) $\phi(z, \tau + \lambda z + \mu) = e^{-2\pi i m(\lambda^2 z + 2\lambda \tau)} \phi(z, \tau)$ for all integers λ, μ ; and
- (iii) ϕ has a Fourier expansion

$$\phi(z, \tau) = \sum_{n \geq 0} \sum_{r^2 \leq 4nm} d(n, r) q^n \zeta^r.$$

A Maass lift [33] is a lifting from the space of Jacobi forms of weight k and index 1 to the space of Siegel modular forms of degree 2.

DEFINITION 4. Let ϕ be a Jacobi cusp form of weight k and index 1 and suppose $\phi(\tau, z) = \sum_{n \geq 0, r \in \mathbb{Z}} d(n, r) q^n \zeta^r$ is its Fourier expansion. Then the Maass lift of ϕ is $\text{Maass}(\phi)$ a Siegel modular form of degree 2 and weight k given by

$$\text{Maass}(\phi) \left(\begin{pmatrix} \tau & z \\ z & w \end{pmatrix} \right) = \sum_{a,b,c} \left(\sum_{\delta | (a,b,c)} \delta^{k-1} d\left(\frac{ac}{\delta^2}, \frac{b}{\delta}\right) \right) q^a \zeta^b q'^c.$$

We observe that $\text{Maass}(\phi)$ corresponds to a formal Siegel modular form C and we compute C coefficient by coefficient:

$$C_\phi([a, b, c]) = \sum_{\delta | (a,b,c)} \delta^{k-1} d\left(\frac{ac}{\delta^2}, \frac{b}{\delta}\right),$$

where $d(n, r)$ are the coefficients of the Jacobi form ϕ .

A map from elliptic modular forms of level 1 to Jacobi forms can be found in [50]; we note that this is not the Saito-Kurokawa lift as it is not Hecke invariant. We summarize the result here. Let $\eta := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ and

$$A := \eta^{-6} \sum_{\substack{r, s \in \mathbb{Z}, r \not\equiv s \pmod{2}}} s^2 (-1)^r q^{(s^2 + r^2)/4} \zeta^r, \text{ and} \\ B := \eta^{-6} \sum_{\substack{r, s \in \mathbb{Z}, r \not\equiv s \pmod{2}}} (-1)^r q^{(s^2 + r^2)/4} \zeta^r$$

For f a degree 1, weight k modular form and g and degree 1 and weight $k + 2$ cusp form, the map, $I : (f, g) \mapsto \frac{k}{2} f A - (q \frac{d}{dq} f) B + g B$ is a Jacobi form of weight k and index 1.

The composition $\text{Maass} \circ I$ yields explicit formulas for the Fourier coefficients of the forms in the Maass spezialsschaar, i.e. in the image of the Maass lift, in terms of the Fourier coefficients of elliptic modular forms of level 1. The formulas for the Fourier coefficients of the Siegel Eisenstein series in Section 3.1 are special cases of this construction.

3.4. Theta Series

Let q be a positive definite quadratic form on a lattice L of rank n and $b(v, w) := q(v + w) - q(v) - q(w)$ be its associated bilinear form. We assume $q(L) \subseteq \mathbb{Z}$. The theta series of degree 2 associated to L can be defined as

$$\theta_L^{(2)}(Z) := \sum_{(v, w) \in L \times L} e(\text{tr}(Z M_{[v, w]}))$$

where $M_{[v, w]} := M_f$ for the binary quadratic form $f = q(xv + yw)$. It is well known that this series is a (scalar valued) Siegel modular form of degree 2 and weight $n/2$, for the level of the lattice and a quadratic character related to the determinant of the bilinear form.

The Fourier coefficients of $\theta_L^{(2)}$ are related to representation numbers of binary quadratic forms by q . Namely,

$$C(f) = \# \{(v, w) \in L \times L : f = q(xv + yw)\}. \quad (3.3)$$

The standard properties of representation numbers imply that $C(A \cdot f) = C(f)$ for $A \in \text{GL}(2, \mathbb{Z})$. So, in particular, a theta series is a formal Siegel modular form with trivial formal character.

The following algorithm computes the Fourier coefficients $C(f)$ of a theta series of degree 2, for $\text{GL}(2, \mathbb{Z})$ -reduced binary quadratic forms f .

ALGORITHM 1. Compute the Fourier coefficients $C(f)$ of $\theta_L^{(2)}$ for $0 > \text{disc} f \geq -X$ as well as $C([0, 0, c])$ for $c \leq c_{\max}$ where c_{\max} is $\lfloor \frac{X+1}{4} \rfloor$.

1. [Precompute] For $1 \leq i \leq c_{\max}$, compute $V_i := \{v \in L : q(v) = i\}$ and $\tilde{V}_i \subset V_i$ such that for every vector $v \in V_i$, exactly one of v or $-v$ is in \tilde{V}_i .
2. [Compute the singular coefficients] $C([0, 0, 0]) = 1$ and for $1 \leq i \leq c_{\max}$ set $C([0, 0, i]) = 2 \cdot \#\tilde{V}_i$.
3. [Iterate through all $\text{GL}(2, \mathbb{Z})$ -reduced forms] For each $1 \leq a \leq \lfloor \sqrt{\frac{X}{3}} \rfloor$ and for each $a \leq c \leq \lfloor \frac{a^2 + X}{4a} \rfloor$, look up V_a and V_c .
4. [Iterate through vectors of shortest length] For $v \in V_a$, compute the linear form $l(w) = b(v, w)$. For $w \in V_c$, compute $b = l(w)$ and $\lambda(b) := \begin{cases} 2 & \text{if } b \neq 0, \\ 4 & \text{if } b = 0. \end{cases}$

5. [Compute coefficient $C[a, b, c]]$ If $b \leq a$ and $b^2 - 4ac \geq -X$, then $C[(a, b, c)] = C[(a, b, c)] + \lambda(b)$.
6. [Done] Return the map $C(f)$.

The algorithm is implemented in Sage a little differently (in particular the loops are in a different order in order to avoid computing things we do not really need to compute).

3.5. Theta constants

As we will see below in Section 4, being able to compute the products of theta constants is a very useful way to compute particular modular forms. For $a, b \in \{0, 1\}^2$ (column vectors), define the theta constant $\theta_{a,b}$ as

$$\theta_{a,b}(Z) = \sum_{\substack{\ell \in \mathbb{Z}^2 \\ \ell \equiv a \pmod{2}}} e\left(\frac{1}{8}Z[\ell] + \frac{1}{4}{}^t\ell b\right) \quad (3.4)$$

where we use $A[B] = {}^tBAB$. It is easy to see that $\theta_{a,b}$ is identically zero if tab is odd. We note that an individual theta constant is not, in general, a formal Siegel modular form as can be seen from the following lemma:

LEMMA 3.1. *For $A \in \mathrm{GL}(2, \mathbb{Z})$, $f \in Q$, we have:*

$$C_{a,b}(A \cdot f) = C_{A^{-1}a, {}^tAb}(f).$$

Note for example, that $\theta_{0000}(Z)$ and $\theta_{0010}(Z) + \theta_{0001}(Z) + \theta_{0011}(Z)$ are formal Siegel modular forms but $\theta_{0010}(Z)$ is not.

We have the following result that allows for the computation of the coefficients of a single theta constant. As noted above, this form is not necessarily a formal Siegel modular form.

PROPOSITION 3.2. *Assume that tab is even. Then the map $C_{a,b} : Q \rightarrow \mathbb{Z}$ given by*

$$C_{a,b}(f) = \sum_{\substack{\ell \equiv a \pmod{2} \\ \ell^2 = f}} (-1)^{\ell(b)/2}$$

describes the coefficients of $\theta_{a,b}(8Z)$. Here Q is as defined in Section 2.2, $a = {}^t(a_1, a_2)$ is identified with the linear polynomial $a_1X + a_2Y$ and the sum is over all linear forms ℓ in $\mathbb{Z}[X, Y]$ which are congruent to $a_1X + a_2Y \pmod{2}$ and whose square equals f .

The claimed formula becomes obvious if one writes $\ell = {}^t(\ell_1, \ell_2)$ and $Z[\ell] = \ell_1^2\tau + 2\ell_1\ell_2z + \ell_2^2\tau'$.

Particular Siegel modular forms are often described as an algebraic combination of theta constants. The following proposition allows for the computation of the Fourier coefficients of such forms.

PROPOSITION 3.3. *Let*

$$F(Z) := \sum_{i=1}^n \alpha_i \prod_{j=1}^d \theta_{a_{ij}, b_{ij}}(Z),$$

where we assume that all ${}^t a_{ij} b_{ij}$ are even. Then the map $C : Q \rightarrow \mathbb{Z}$ associated to $F(8Z)$ is given by

$$C(f) = \sum_{i=1}^{n-1} \alpha_i \sum_{\substack{\ell_1, \dots, \ell_d \equiv a_{i1}, \dots, a_{id} \pmod{2} \\ \ell_1^2 + \dots + \ell_d^2 = f}} (-1)^{(\ell_1(b_{i1}) + \dots + \ell_d(b_{id}))/2}.$$

The C in the previous two propositions can be formal Siegel modular forms. In that case we only compute the values of C at $\mathrm{GL}(2, \mathbb{Z})$ -reduced quadratic forms in Q , taking advantage of the extra structure a formal Siegel modular form has. We note, also, that whether or not C is a formal Siegel modular form has to be determined before the computations are carried out.

The above definitions and propositions can be generalized to other levels. One should change these results in the expected way: e.g., the sums are mod N , the forms $\theta_{a,b,N}(2N^2Z)$ are formal Siegel modular forms, etc. In particular, we write

$$\theta_{N,a,b}(Z) = \sum_{\substack{\ell \in \mathbb{Z}^2 \\ \ell \equiv a \pmod{N}}} e \left(\frac{1}{2N^2} Z[\ell] + \frac{1}{N^2} {}^t \ell b \right), \quad (3.5)$$

where again we use $A[B] = {}^t BAB$.

REMARK 1. A useful observation is that one can always take $b = 0$ to get the space spanned by theta constants. In particular, the span of $\theta_{a,b,N}(Z)$ is the same as the span of $\theta_{a,0,N}(Z)$.

3.6. Vector-valued theta series

Following [25] we can define vector-valued theta series. Take a rank n even unimodular lattice L , let $P \in \mathbb{C}[X, Y]_j$ be pluriharmonic (i.e., for every $A \in \mathrm{GL}(2, \mathbb{Z})$, $P(AX) = \det(A)^k \mathrm{Sym}_j(A)P(X)$). Then we define

$$\theta_{L,P}(Z) = \sum_{x,y \in L} P \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) e^{\pi i((x,x)\tau + (x,y)z + (y,y)\tau')}.$$

One can show that $\theta_{L,P} \in M_{n/2+k,j}(\Gamma)$. After collecting terms, $\theta_{L,P}$ is evidently a formal Siegel modular form because of the invariance of the pluriharmonic polynomials that make up the coefficients.

To describe the generators of a certain space of vector-valued Siegel modular forms, it is useful to define the following two pluriharmonic polynomials. Let $a, b \in \mathbb{C}^n$ be such that their inner products $(a, a) = (a, b) = (b, b) = 0$ (here the inner product of a and b is given by $\sum a_i b_i$, without complex conjugation). Consider the two polynomials

$$P_a^{(j)} = \sum_{\nu=0}^k (x, a)^{(j-\nu)} (y, a)^\nu X^\nu Y^{j-\nu}$$

$$P_{a,b,k}^{(j)} = \sum_{\nu=0}^k (x, a)^{(j-\nu)} (y, a)^\nu \det \begin{pmatrix} (x, a) & (y, a) \\ (x, b) & (y, b) \end{pmatrix}^k X^\nu Y^{j-\nu}.$$

With this notation, $\theta_{oL, P_a^{(j)}} \in M_{d/2,j}(\Gamma)$, $\theta_{L, P_{a,b,k}^{(j)}} \in M_{d/2+k,j}(\Gamma)$.

In order to compute these theta series, one needs to adjust Algorithm 1 appropriately.

3.7. Wronskians

We recall a useful construction in [2]. A Siegel modular form of degree 2 is a function in three variables, τ , z and τ' . Suppose we have 4 algebraically independent Siegel modular forms

F_1, F_2, F_3, F_4 of weights k_1, k_2, k_3, k_4 , respectively. Then the function

$$\{F_1, F_2, F_3, F_4\} := \begin{vmatrix} k_1 F_1 & k_2 F_2 & k_3 F_3 & k_4 F_4 \\ \frac{\partial F_1}{\partial \tau} & \frac{\partial F_2}{\partial \tau} & \frac{\partial F_3}{\partial \tau} & \frac{\partial F_4}{\partial \tau} \\ \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial z} & \frac{\partial F_3}{\partial z} & \frac{\partial F_4}{\partial z} \\ \frac{\partial F_1}{\partial \tau'} & \frac{\partial F_2}{\partial \tau'} & \frac{\partial F_3}{\partial \tau'} & \frac{\partial F_4}{\partial \tau'} \end{vmatrix} \quad (3.6)$$

is a nonzero Siegel cusp form of weight $k_1 + k_2 + k_3 + k_4 + 3$. See [2, Prop. 2.1] for more details and for the statement for arbitrary degree.

Computationally speaking, computing the determinant $\{F_1, \dots, F_4\}$ in the straightforward way is very expensive. Many products (36, computing the determinant in the naive way) of multivariate power series would need to be computed in order to find F . However, using basic properties of the determinant, one deduces

PROPOSITION 3.4. *With notation as above, we have*

$$F = \{F_1, F_2, F_3, F_4\} = \sum_{\substack{T_1=[a_1, b_1, c_1] \geq 0 \\ T_2=[a_2, b_2, c_2] \geq 0 \\ T_3=[a_3, b_3, c_3] > 0 \\ T_4=[a_4, b_4, c_4] > 0}} C_{F_1}(T_1) C_{F_2}(T_2) C_{F_3}(T_3) C_{F_4}(T_4) \\ \times \begin{vmatrix} k_1 & k_2 & k_3 & k_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} e(\sum a_i \tau + \sum b_i z + \sum c_i \tau'). \quad (3.7)$$

This is the basis for Algorithm 2, below. The subtle part in the algorithm is determining the bounds of the coefficients of the T_i . These bounds have to be as tight as possible, since even a slight inaccuracy leads to subsequent iteration of many unnecessary Fourier indices.

Proof of Proposition 3.4. Equation (3.7) follows from writing (4.1) in the form

$$F(Z) = \det \left(\sum_{T_1=[a_1, b_1, c_1]} C_{F_1}(T_1) \begin{pmatrix} k_1 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} \right) e(a_1 \tau + b_1 z + c_1 \tau'), \\ \sum_{T_2=[a_2, b_2, c_2]} C_{F_2}(T_2) \begin{pmatrix} k_2 \\ a_2 \\ b_2 \\ c_2 \end{pmatrix} e(a_2 \tau + b_2 z + c_2 \tau'), \dots$$

and then using that the determinant function is multilinear. \square

The algorithm below to compute $\{F_1, \dots, F_4\}$ can be changed in the obvious way to work for arbitrary degree.

ALGORITHM 2. Compute the Fourier coefficients $C(f)$ of $\{F_1, \dots, F_4\}$ for $0 > \text{disc } f \geq -X$ as well as $C([0, 0, c])$ where $c \leq \lfloor \frac{X+1}{4} \rfloor$.

1. [Precompute quadratic forms and modular forms] Compute $Q_X := \{f \in Q : 0 \geq \text{disc } f \geq -X\} \cup \{f = [0, 0, c] \in Q : c \leq \lfloor \frac{X+1}{4} \rfloor\}$; compute coefficients $C_{F_1}(f), C_{F_2}(f), C_{F_3}(f), C_{F_4}(f)$ for all $f \in Q_X$.

2. [Compute the coefficient at f] For each 4-tuple (T_1, T_2, T_3, T_4) where $T_i \in Q_X$ and $\sum T_i = f$, set

$$C(f) = \sum_{\substack{T_1=[a_1, b_1, c_1] \geq 0 \\ T_2=[a_2, b_2, c_2] \geq 0 \\ T_3=[a_3, b_3, c_3] \geq 0 \\ T_4=[a_4, b_4, c_4] \geq 0}} C_{F_1}(T_1) C_{F_2}(T_2) C_{F_3}(T_3) C_{F_4}(T_4) \times \begin{vmatrix} k_1 & k_2 & k_3 & k_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}.$$

3. [Done] Return the map $C(f)$.

3.8. Rankin-Cohen Brackets

The attempt to construct a modular form as a linear combination of products of derivatives of two modular forms is known as the Rankin-Cohen construction. A complete investigation of this and related constructions has been given by Ibukiyama in [24]. See also [8, 11, 34] for characterizations of various Rankin-Cohen brackets.

3.8.1. Satoh Bracket The Satoh bracket is a special case of this general construction. Satoh examined the case $\rho = \text{sym}_2 \otimes \det^k$. Suppose $F \in M_k(\Gamma)$ and $G \in M_{k'}(\Gamma)$. Then

$$[F, G]_{\text{sym}_2} = \frac{1}{2\pi i} \left(\frac{1}{k} G \partial_Z F - \frac{1}{k'} F \partial_Z G \right) \in M_{k+k', 2}(\Gamma), \quad (3.8)$$

where $\partial_Z = \begin{pmatrix} \partial_{Z_{11}} & 1/2 \partial_{Z_{12}} \\ 1/2 \partial_{Z_{12}} & \partial_{Z_{22}} \end{pmatrix}$.

In order to implement the Satoh bracket in our package, we need first verify that the forms $\partial_Z F$ and $\partial_Z G$ are formal Siegel modular forms, even if they are not Siegel modular forms.

LEMMA 3.5. *Let F be a modular form. Then $\partial_Z F$ as defined above is a formal Siegel modular form.*

Proof. Let $F_Z := \partial_Z F$ and $C_{F_Z} : Q \rightarrow \mathbb{C}[X, Y]_j$ be such that $C_{F_Z}([a, b, c])$ takes the coefficient of F_Z at $[a, b, c]$ as its value. We need to show that C_{F_Z} is formal; i.e., that $C_{F_Z}(A \cdot [a, b, c]) = (\det A)^k A \cdot C_{F_Z}([a, b, c])$.

A straightforward calculation shows that $C_{F_Z}([(a, b, c)]) = C_F([a, b, c])(aX^2 + bXY + cY^2)$. Now

$$\begin{aligned} C_{F_Z}(A \cdot [a, b, c]) &= C_F(A \cdot [a, b, c]) A \cdot (aX^2 + bXY + cY^2) \\ &= (\det A)^k C_F([a, b, c]) A \cdot (aX^2 + bXY + cY^2) \\ &= (\det A)^k A \cdot (C_F([a, b, c])(aX^2 + bXY + cY^2)) \\ &= (\det A)^k A \cdot C_{F_Z}([a, b, c]). \end{aligned}$$

□

Because of Lemma 3.5, we can compute the Satoh bracket as the difference of the product of two formal Siegel modular forms.

Assuming that multiplication of vector valued Fourier expansions is implemented an algorithm can now be formulated as follows.

ALGORITHM 3.

1. For f and g compute the Fourier expansion with coefficients $C_{\partial_Z F}(T) = C(T)(X^2 T_{11} + XY T_{12} + Y^2 T_{22})$.

2. Multiply the Fourier expansions according to formula (3.8).

3.8.2. Ibukiyama's Brackets There are constructions found in [25] that are analogous to the Satoh bracket but whose images are in $M_{k,4}(\Gamma)$ and $M_{k,6}(\Gamma)$. To give an indication of what they look like, we provide an example: let $F \in M_k(\Gamma)$ and $G \in M_{k'}(\Gamma)$. Then if k is even,

$$[F, G]_{\text{sym}_4} = \frac{k'(k'+1)}{2} G \times \begin{pmatrix} \frac{\partial^2 F}{\partial \tau^2} \\ 2 \frac{\partial^2 F}{\partial \tau \partial z} \\ \frac{\partial^2 F}{\partial z^2} + 2 \frac{\partial^2 F}{\partial \tau \partial \tau'} \\ 2 \frac{\partial^2 F}{\partial \tau' \partial z} \\ \frac{\partial^2 F}{\partial \tau'^2} \end{pmatrix} - \\ (k'+1)(k+1) \begin{pmatrix} \frac{\partial F}{\partial \tau} \frac{\partial G}{\partial \tau} + \frac{\partial F}{\partial \tau'} \frac{\partial G}{\partial \tau'} \\ \frac{\partial F}{\partial \tau} \frac{\partial G}{\partial \tau'} + \frac{\partial F}{\partial \tau'} \frac{\partial G}{\partial \tau} \\ \frac{\partial F}{\partial \tau} \frac{\partial G}{\partial z} + \frac{\partial F}{\partial \tau'} \frac{\partial G}{\partial z} \\ \frac{\partial F}{\partial \tau'} \frac{\partial G}{\partial \tau} + \frac{\partial F}{\partial \tau} \frac{\partial G}{\partial \tau'} \\ \frac{\partial F}{\partial \tau'} \frac{\partial G}{\partial \tau'} + \frac{\partial F}{\partial \tau} \frac{\partial G}{\partial \tau} \end{pmatrix} + \frac{k(k+1)}{2} \begin{pmatrix} \frac{\partial^2 G}{\partial \tau^2} \\ 2 \frac{\partial^2 G}{\partial \tau \partial z} \\ \frac{\partial^2 G}{\partial z^2} + 2 \frac{\partial^2 G}{\partial \tau \partial \tau'} \\ 2 \frac{\partial^2 G}{\partial \tau' \partial z} \\ \frac{\partial^2 G}{\partial \tau'^2} \end{pmatrix}.$$

3.9. Borcherds Products

In [5], Borcherds introduced a multiplicative lift from whose image can be a Siegel modular form of degree 2. The Fourier expansions of these Borcherds products can be computed in polynomial time [18]. We refer the interested reader to the latter article, co-authored by the first author, for more details.

3.10. Further remarks

3.10.1. Restriction method We are going to discuss the approach that was used to obtain the results in [43]. In this work Siegel modular forms of degree 4 were considered, but in principle the algorithm can also be used to obtain Fourier expansions of forms for high level and degree 2. The idea is as follows. Suppose we restrict a Siegel modular form $f : \mathbb{H}_2 \rightarrow \mathbb{C}$ to the subspace $\phi_{s,\zeta} : \mathbb{H}_1 \rightarrow \mathbb{H}_2$, $\tau \mapsto s\tau + \zeta$ for some positive definite symmetric $s \in M(2, \mathbb{Z})$ and some symmetric $\zeta \in M(2, \mathbb{Q})$. This will lead to an elliptic modular form whose Fourier expansion can be deduced from the Fourier expansion of f . Since many cusps for $f \circ \phi_{s,\zeta}$ can be identified in \mathbb{H}_2 , there are non trivial obstructions to the Fourier expansion of f . In [43], where all cusps identified with each other, this provided enough restrictions to actually compute the initial Fourier expansions of a basis for a space of fixed weight. They could not give a proof that this method will work for all Fourier indices and all weights. In the case of level subgroups a further difficulty enters. There will be cusps that cannot be identified with each other in \mathbb{H}_2 .

3.10.2. Poor-Yuen Conjecture The Fourier-Jacobi expansion of a Siegel modular form can be used to describe a Fourier expansion. A formal Fourier-Jacobi expansion is a formal series $\sum f_m q^m$ of Jacobi forms f_m that have level m . Chris Poor and David Yuen conjectured that a formal Fourier-Jacobi expansion is the Fourier-Jacobi expansion a Siegel modular form for the full modular group if and only if the Fourier coefficients of the Jacobi forms satisfy the condition $f_m(n, r) = f_n(m, r)$. It was recently announced by Poor and Yuen that have proved this result.

3.10.3. Invariant method and Igusa's theorem on theta constants We fix a level N . For $a, b \in \{0, \dots, N-1\}^2$ (column vectors), define the theta constant $\theta_{N,a,b}$ as in (3.5). It is easy

to see that $\theta_{N,a,b}$ is identically zero if a/N and b/N have half integral entries and ${}^t ab$ is odd. [30] tells us that also the converse holds. There we can also find a proof that they are modular forms with respect to the following subgroup $\Gamma(N, N)$ of $\mathrm{Sp}(2, \mathbb{Z})$:

$$\left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Z}) : \gamma \equiv 1_4 \pmod{N}; A^t B, C^t D \text{ have diagonals divisible by } 2N \right\}.$$

As we will see below in Section 4.2, being able to compute the products of theta constants is a very useful way to describe modular forms.

An important result is

THEOREM 3.6. *Suppose that $\Gamma(N, N) \subseteq \Gamma$ is a congruence subgroup. The graded ring of modular forms for Γ is the integral closure of the ring of those modular forms with respect to Γ that are formed as linear combinations of products of the theta series $\theta_{N,a,b}$.*

This method to compute rings of Siegel modular forms is impractical for large level. Nevertheless its connection to coding theory makes it interesting for general level. This can be found in [47].

3.10.4. The basis problem and theta series The basis problem, as formulated by Eichler [12][13], asks whether the space of elliptic modular forms for fixed weight, level and character can be spanned by θ -series. The affirmative answer in the elliptic case partially generalizes to Siegel modular forms. The most complete treatment can be found in [3], where Böcherer proves the affirmative answer in the case of squarefree level and also gives a brief survey on the current effort in this area. In particular, he proves the following:

THEOREM 3.7. *Assume that $N \in \mathbb{N}$ is squarefree and $k > 5$. Let χ be a character for $\Gamma_0(N)$. Then all cusp forms in $M_k(\Gamma_0(N), \chi)$ are linear combinations of θ -series.*

We also mention the very early treatment in [39]. There Ozeki proves that in the case of the full modular group one might restrict to unimodular lattices if and only if the weight is divisible by 4.

There is a special case of this question. The critical weight for degree n forms is $n/2$; in the case of degree 2 the critical weight is 1. It is expected that these weights behave particularly well with respect to the basis problem. In [54] it was proved that every Siegel modular form of degree 2 and weight 1 is a linear combination of θ -series. At this point it might also be useful to the reader to mention the following results of the third named author. In [22] it was proved that there are no Siegel cusp forms of weight one for any group $\Gamma_0(N)$ of Hecke type.

Any modular form of weight less than the critical weight is called singular. In the case of degree 2 these are the half integral weight forms. Singular weights are well-understood for arbitrary degrees (see [17]). In particular, we know that all weight $\frac{1}{2}$ forms associated to the full modular group only have nonvanishing Fourier coefficients for indefinite indices. Hence, by a theorem of Serre and Stark [49] they are given by θ -series.

4. Rings of Siegel Modular Forms

The problem of determining rings of modular forms has been of great interest. Complete descriptions of some rings are presented in Section 4.2. These results are almost entirely based on a combination of the theory of Borcherds products (see 3.9), θ -series, and the invariant method that we discussed in Section 3.10.3.

Besides these results, there are methods that have been developed for cases where a set of generators has not yet been obtained. Although we focus on the descriptions mentioned above, we discuss some of these other methods in the next few subsections. Indeed, even for moderate levels, rings of Siegel modular forms become complicated. For applications it is sometimes sufficient to obtain a set of modular forms that spans a space of fixed weight. In these cases it is necessary to know the dimensions of these spaces. For this reason we also review known dimension formulas.

4.1. Hecke eigenforms

Hecke eigenforms are the modular forms that are of the greatest arithmetic interest. In order to compute eigenforms, one needs to compute a space of cusp forms, find the matrix for a Hecke operator and from that find the eigenforms. In this section we address issues surrounding this process. First, one needs to know when one has computed a basis for the space and to do this one needs to know the dimension of the space. Second, one needs to carry out the linear algebra computations described above.

4.1.1. Dimension formulas In most cases only formulas for dimensions of spaces of cusp forms are available. This might already be sufficient, since many arithmetic questions can be reduced to this case. In Table 1 we give some dimensions of spaces of Siegel modular forms.

4.1.2. Computing Hecke eigenforms Among all modular forms, the most arithmetically distinguished are the Hecke eigenforms. Fix a space of Siegel modular forms with basis $\{F_1, \dots, F_n\}$. Because the Hecke operators are a commuting family of linear operators, there is a basis $\{G_1, \dots, G_n\}$ for the space made up entirely of simultaneous eigenforms.

The forms G_i are determined computationally as follows. First, determine the matrix representation for the Hecke operator T_2 . Do this by computing the image under T_2 of each basis element F_i . Build a matrix N that is invertible and whose j th row consists of coefficients of F_j at certain indices Q_1, \dots, Q_n . To ensure that N is invertible we pick the indices one at a time, making sure that each choice of index Q_i increases the rank of the matrix. We then construct a matrix M whose j th row consists of coefficients of the image of F_j under T_2 indexed by Q_1, \dots, Q_n . Then the matrix representation of T_2 is $M \cdot N^{-1}$.

Second, one finds the eigenspaces of the matrix for T_2 and uses that to describe the basis G_1, \dots, G_n . Typically only a single member of a given Galois orbit is described. See, for example, [51].

4.2. Particular Rings of scalar-valued Forms

4.2.1. The full Siegel modular group In [27] and [28], the five generators of the ring of Siegel modular forms and level 1 are identified: they are E_4 , E_6 , χ_{10} , χ_{12} , χ_{35} and are of weights 4, 6, 10, 12 and 35. The first two are Eisenstein series and the last three are cusp forms. In [50] an indication of how to compute the four generators of even weight is given. One composes

$M_k(\mathrm{Sp}(2, \mathbb{Z}))$	[27, 50]
$S_k(\Gamma_1(\ell)), N \geq 3, k \geq 5$	[9, 38, 55]
$S_k(\Gamma_0(p)), p \text{ prime}, k \geq 5$	[20]
$S_4(\Gamma_0(p)), p \leq 13 \text{ prime}$	[41]
$S_3(\Gamma_0(p)), p \leq 23 \text{ prime}$	[41]
$S_2(\Gamma_0(p)), p \leq 41 \text{ prime}$	[41]
$S_1(\Gamma_0(N)), N \geq 1$	[22]
$S_{k,j}(\mathrm{Sp}(2, \mathbb{Z})), j \geq 0, k \geq 4$	[53]

TABLE 1. Spaces and references for their dimension formulas

an explicit map from classical to Jacobi forms and the Maass lift from Jacobi forms to Siegel modular forms of degree 2. The implementation in [46] of these four generators follows [50].

In order to get the full ring of Siegel modular forms of level 1 without character, we require a method to compute χ_{35} . In [28], it is given in terms of theta constants. In [19], it is given as a Borcherds product. In [2], it is given as a Wronskian (see Section 3.7):

$$\chi_{35} = \frac{1}{(2\pi i)^3} \{E_4, E_6, \chi_{10}, \chi_{12}\}. \quad (4.1)$$

Here $E_4, E_6, \chi_{10}, \chi_{12}$ are normalized so that $C_{E_4}([0, 0, 0]) = C_{E_6}([0, 0, 0]) = 1$ and $C_{\chi_{10}}([1, 1, 1]) = C_{\chi_{12}}([1, 1, 1]) = 1$.

If one is content with just a basis for the vector space of Siegel modular forms of a fixed even weight, an observation by the first author is useful and would make, if true in general, the computation of such a basis significantly less expensive. In particular, in [45] he has observed that up to weight 172 one can span the space of weight k Siegel modular forms of degree 2 via forms that are the sum of products of no more than two Saito-Kurokawa lifts; see Section 3.3 for more details on Saito-Kurokawa lifts. Since computing a Saito-Kurokawa lift is rather straightforward, multiplying a pair of Siegel modular forms is not terribly expensive, and adding many Siegel modular forms is very fast, this would represent a significant speed-up over what is done now.

4.2.2. Congruence subgroups of small level In papers by Ibukiyama and his coauthors, the generators of some rings of modular forms of small level have been identified. Let $\Gamma_0(\ell)$ be the Klingen subgroup of level ℓ . For a subgroup Γ we use $M_*(\Gamma, \chi)$ for the ring of Siegel modular forms of degree 2 with respect to the group Γ and with character χ (we omit the χ if it is trivial).

The groups whose rings of modular forms are known to be described in terms of generators are listed in Table 2

Group and Character	Reference
$\Gamma_0(2)$	[2]
$\Gamma_0^{\psi_3}(3) := \{\gamma \in \Gamma_0(3) : \psi_3(\gamma) = 1\}$ where $\psi_3\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) := \left(\frac{-3}{\det D}\right)$, for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(3)$	[2]
$\Gamma_0^{\psi_4}(4) := \{\gamma \in \Gamma_0(4) : \psi_4(\gamma) = 1\}$ where $\psi_4\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) := \left(\frac{-1}{\det D}\right)$, for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4)$	[21]

TABLE 2. Subgroups Γ for which the generators of the ring $M_*(\Gamma)$ are known.

4.2.3. Siegel modular forms of degree 2 and level 2 Consider the ring $M_*(\Gamma_0(2))$. In [2] it is shown that the ring has 5 generators:

$$\begin{aligned} X &:= (\theta_{(0,0),(0,0)}^4 + \theta_{(0,0),(0,1)}^4 + \theta_{(0,0),(1,0)}^4 + \theta_{(0,0),(1,1)}^4)/4 \\ Y &:= (\theta_{(0,0),(0,0)}\theta_{(0,0),(0,1)}\theta_{(0,0),(1,0)}\theta_{(0,0),(1,1)})^2 \\ Z &:= (\theta_{(0,1),(0,0)}^4 - \theta_{(0,1),(1,0)}^4)^2/16384 \\ K &:= (\theta_{(0,1),(0,0)}\theta_{(0,1),(1,0)}\theta_{(1,0),(0,0)}\theta_{(1,0),(0,1)}\theta_{(1,1),(0,0)}\theta_{(1,1),(1,1)})^2/4096 \\ \chi_{19} &= \{X, Y, Z, K\} \end{aligned}$$

The generators are, respectively, of weights 2, 4, 4, 6 and 19. The first four are given in terms of theta constants and the last one is given as a Wronskian.

4.2.4. *Siegel modular forms of degree 2 and level 3* The ring $M_*(\Gamma_0^{\psi(3)}(3))$ has 5 generators. Consider the quadratic forms

$$\begin{aligned} A_2 &:= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \\ E_6 &:= \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & -1 \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \\ E_6^* &:= 3E_6^{-1}, \\ S_4 &:= \begin{pmatrix} 1 & & 3/2 & \\ 3/2 & 1 & & 3/2 \\ & & 3 & \\ 3/2 & & & 3 \end{pmatrix}, \text{ and} \\ Q(x, y) &:= \begin{pmatrix} {}^t x S x & {}^t x S y \\ {}^t y S x & {}^t y S y \end{pmatrix}. \end{aligned}$$

The modular forms of note are

$$\begin{aligned} \alpha_1 &:= \theta_{A_2}, \\ \beta_3 &:= \theta_{E_6} - 10\alpha_1^3 + 9\theta_{E_6^*}, \\ \delta_3 &:= \theta_{E_6} - 9\theta_{E_6^*}, \\ \gamma_4(Z) &:= \sum_{\substack{(x_1, \dots, x_4) \in \mathbb{Z}^4 \\ (y_1, \dots, y_4) \in \mathbb{Z}^4}} (c^2 - d^2)e(\text{tr}(Q(x, y)Z)) \text{ where} \\ c &:= (x_1 y_3 - x_3 y_1) + (x_2 y_4 - x_4 y_2) \text{ and} \\ d &:= (x_1 y_4 - x_4 y_1) + (x_3 y_2 - x_2 y_3) \\ \chi_{14} &:= \{\alpha_1, \beta_3, \delta_3, \gamma_4\}. \end{aligned}$$

We observe that these forms, respectively, have weights 1, 3, 3, 4, 14. With this notation, then,

$$M_*(\Gamma_0^{\psi_3}(3)) = \mathbb{C}[\alpha_1, \beta_3, \delta_3, \gamma_4] \oplus \chi_{14} \mathbb{C}[\alpha_1, \beta_3, \delta_3, \gamma_4].$$

4.2.5. *Siegel modular forms of degree 2 and level 4* Now, we consider $\Gamma_0^{\psi_4}(4)$. First, as in [28], let

$$\begin{aligned} \chi_5 &:= \theta_{(0,0),(0,0)}(Z)\theta_{(0,0),(0,1)}(Z)\theta_{(0,0),(1,0)}(Z)\theta_{(0,0),(1,1)}(Z)\theta_{(0,1),(0,0)}(Z) \\ &\quad \times \theta_{(0,1),(1,0)}(Z)\theta_{(1,0),(0,0)}(Z)\theta_{(1,0),(0,1)}(Z)\theta_{(1,1),(0,0)}(Z)\theta_{(1,1),(1,1)}(Z). \end{aligned}$$

Then, as in [2], let

$$\begin{aligned} f_{1/2} &:= \theta_{(0,0),(0,0)}(2Z) \\ f_1 &:= f_{1/2}^2 \\ g_2 &:= \theta_{(0,0),(0,0)}(2Z)^4 + \theta_{(0,1),(0,0)}(2Z)^4 + \theta_{(1,0),(0,0)}(2Z)^4 + \theta_{(1,1),(0,0)}(2Z)^4 \\ h_2 &:= \theta_{(0,0),(0,0)}(2Z)^4 + \theta_{(0,0),(0,1)}(2Z)^4 + \theta_{(0,0),(1,0)}(2Z)^4 + \theta_{(0,0),(1,1)}(2Z)^4 \\ f_3 &:= (\theta_{(0,0),(0,1)}(2Z)\theta_{(0,0),(1,0)}(2Z)\theta_{(0,0),(1,1)}(2Z))^2 \\ \chi_{11} &:= \{f_1, g_2, h_2, f_3\}. \end{aligned}$$

Then $M_*(\Gamma_0^{\psi_4}(4)) = \mathbb{C}[f_1, g_2, h_2, f_3, \chi_{11}]$.

4.3. Particular spaces of vector valued Forms

The ring of all vector valued Siegel modular forms $\bigoplus_{k,j} M_{k,j}(\Gamma)$ is not finitely generated. For this reason the vector valued weight j is usually fixed. The resulting module is finitely generated over $\bigoplus_k M_k(\Gamma)$.

Before we summarize the results on such modules of vector valued modular forms, we remark that vector valued Siegel modular forms are always cuspidal. This follows from a much more general result by Weissauer [54], that covers Siegel modular forms of all degrees.

We summarize the spaces that we can compute in Table 3.

4.3.1. Satoh's Theorem By Satoh [48] we know that $\bigoplus_k M_{k,2}(\Gamma)$ is generated by 6 elements, all of which can be expressed in terms of Satoh brackets (see Section 3.8.1). More precisely, he shows that

$$\begin{aligned} M_{k,2}(\Gamma) = & [E_4, E_6] \cdot M_{k-10}(\Gamma) \oplus [E_4, \chi_{10}] \cdot M_{k-14}(\Gamma) \oplus \\ & [E_4, \chi_{12}] \cdot M_{k-16}(\Gamma) \oplus [E_6, \chi_{10}] \cdot \mathbb{C}[E_6, \chi_{10}, \chi_{12}]_{k-16} \oplus \\ & [E_6, \chi_{12}] \cdot \mathbb{C}[E_6, \chi_{10}, \chi_{12}]_{k-18} \oplus [\chi_{10}, \chi_{12}] \cdot \mathbb{C}[\chi_{10}, \chi_{12}]_{k-22}. \end{aligned}$$

By $\mathbb{C}[A_1, \dots, A_n]_k$ we mean the module of weight k modular forms that can be expressed in terms of generators A_1, \dots, A_n .

4.3.2. Ibukiyama's Theorems In [25], generators for the rings $M_{k,4}$ and $M_{k,6}$ are given. The ring of forms of weight $(k, 4)$ is generated by 10 forms that are defined in terms of differential operators similar to the Satoh bracket mentioned above. The ring of forms of weight $(k, 6)$ are generated by a Klingen Eisenstein series, two theta series with pluriharmonics (see Section 3.6 for more information), and four forms that are defined via differential operators.

5. A particular implementation

In this section we describe our implementation of a package that can handle a wide variety of Siegel modular forms. In particular, one can multiply Satoh brackets by scalar-valued Siegel modular forms and one can multiply products of theta constants by other similar products. In short, our package allows for the multiplication of any two formal Siegel modular forms (even if the formal Siegel modular forms do not actually correspond to actual Siegel modular forms). Of particular note is that our implementation handles Siegel modular forms compactly and efficiently.

5.1. An implementation in Sage

A Siegel modular form is implemented in the Sage package [46] as a formal Siegel modular form, i.e. as a map $C : Q \rightarrow R$, where R is a module (or ring) with a $\mathrm{GL}(2, \mathbb{Z})$ -action, such that

$$A^{-1}.C(A.f) = C(f)$$

for all A in $\mathrm{GL}(2, \mathbb{Z})$ and all f in Q . Such a map can be readily implemented via a class `SiegelModularFormsClass` encapsulating a (Python-)dictionary whose keys are the $\mathrm{GL}(2, \mathbb{Z})$ -reduced quadratic forms below a certain bound. The methods of this class include multiplication, addition, Rankin-Cohen brackets, Hecke operators, etc. The bottlenecks for an effective implementation of these methods are multiplication and reduction of integral binary quadratic forms.

Group and Weights	Reference
$\Gamma, j = 2, k \in \mathbb{Z}$	[48]
$\Gamma, j = 4, 6, k \in \mathbb{Z}$	[25]

TABLE 3. Spaces of vector valued Siegel modular forms whose generators are known.

REMARK 2. We observe that our implementation of Siegel modular forms in Sage allows for the precision of a Siegel modular form F to be described in a number of ways. When we say F has discriminant precision X (here X is a positive integer), we mean that the keys of the dictionary mentioned above are the $\mathrm{GL}(2, \mathbb{Z})$ -reduced quadratic forms of discriminant greater than $-X$.

There are a handful of other ways to describe the precision of a Siegel modular form (e.g., trace and box precisions). In [46] we include a Siegel modular form precision class that handles translations between the various kinds of precision.

For many theoretical purposes, it is best to describe the precision of a Siegel modular form in terms of dyadic trace [40, 41, 42, 43, 44]. Poor and Yuen have exploited this idea to prove, among other results, a nonvanishing theorem that can be thought of as a generalization of Sturm's bound for classical modular forms.

5.2. Multiplication

The multiplication of Siegel modular forms is expensive because it involves the evaluation of triple sums, whereas the reduction has to be applied in almost every call to a method of the class `SiegelModularForms_class` since its instances only store coefficients for reduced forms.

The obvious algorithm for the reduction is a variant of the Euclidean algorithm: applying repeatedly to a form f the two steps $[a, b, c] \mapsto [a, b\%a, c]$ (where $b\%a$ denote the Euclidean remainder of b modulo a) and $[a, b, c] \mapsto [c, b, a]$ yields eventually the reduced form which is $\mathrm{GL}(2, \mathbb{Z})$ -equivalent to f .

For a fast multiplication it is useful to provide separate implementations according to the types of the values of the formal Siegel modular forms. Operations on integer valued Siegel modular forms can clearly be handled faster than on those taking values, for example, in a polynomial ring over a number field.

We point out that by “fast multiplication” we do not mean Karatsuba multiplication or the like. There are Karatsuba-like algorithms for multivariate power series rings, but they do not take advantage of the invariance of the coefficients for the Siegel modular form. The (potential) gains won by implementing a Siegel modular form as a multivariate power series rings that has a Karatsuba-like multiplication are outweighed by the great number more coefficients such an object would have to store and the number of redundant multiplications that would have to be done. There is currently no known Karatsuba-type algorithm that respects the invariance of the coefficients of a Siegel modular form, but this will be investigated in future work by the first author.

As a final remark we note that the set of all formal Siegel modular forms for a given coefficient ring R , i.e. the set of formal Siegel modular forms $C : Q \rightarrow R(\chi)$ (χ a linear character of $\mathrm{GL}(2, \mathbb{Z})$), is naturally equipped with a multiplication. Namely, for two such formal Siegel modular forms $C_1 : Q \rightarrow R(\chi_1)$ and $C_2 : Q \rightarrow R(\chi_2)$ the map C defined by

$$C(f) = \sum_{\substack{f_1, f_2 \in Q \\ f = f_1 + f_2}} C_1(f_1)C_2(f_2)$$

defines a formal Siegel modular form $C : Q \rightarrow R(\chi_1\chi_2)$. Note that we would like to allow different characters since we would like to be able to multiply, for example, a formal Siegel modular form corresponding to even weight scalar-valued Siegel modular form (and hence with trivial character) with formal Siegel modular forms corresponding to scalar-valued Siegel modular forms of odd weight (and hence with character \det).

However, if χ_1 is different from χ_2 there is no natural sum of C_1 and C_2 . On the other hand in our Sage implementation it is desirable to view formal Siegel modular forms as elements of an ambient algebra. This allows for an implementation which is consistent with Sage's internal

coercion system, which can take then over the necessary coercion steps to multiply (and add) formal Siegel modular forms with different coefficient rings if the coefficients can be coerced to a common ring.

To view formal Siegel modular of a given coefficient ring as elements of an algebra we proceed as follows. Let Ξ be the group of linear characters of $\mathrm{GL}(2, \mathbb{Z})$. We view a formal Siegel modular form with values in $R(\chi)$ as a formal Siegel modular form with values in the group ring $R[\Xi]$ equipped with the $\mathrm{GL}(2, \mathbb{Z})$ -action

$$(A, \sum_{\chi \in \Xi} c(\chi) e_\chi) \mapsto \sum_{\chi \in \Xi} \chi(A) A.c(\chi) e_\chi$$

via the natural embedding $R(\chi) \rightarrow R[\Xi]$ which maps r to re_χ . Here e_χ runs through the natural basis of the group ring $R[\Xi]$. Formal Siegel modular forms with values in $R[\Xi]$ then naturally form an algebra over the base ring of R (which is either R if the action of $\mathrm{GL}(2, \mathbb{Z})$ on R is trivial or the base ring R' of R if R is a polynomial ring in two variables over R' equipped with the natural $\mathrm{GL}(2, \mathbb{Z})$ -action) via the Cauchy product mentioned above and the obvious addition.

5.3. Data

At [51], one can find data for scalar and vector valued Siegel modular forms of level 1. In the future, similar data for Siegel modular forms of level greater than 1 will be posted. In particular, it is our goal to have data for all the rings and spaces listed in Tables 3 and 2.

6. Generalizations

A class similar to the `SiegelModularForms_class` would work also for other types of automorphic forms like Hilbert modular forms, Siegel modular forms of higher degree, orthogonal modular forms, etc. We formulate here in an abstract way the kind of object should be implemented to treat all kinds of higher rank automorphic forms in a unified way.

Let G be a group, let R be a module commutative ring and M be a monoid, both equipped with a G -action $(g, r) \mapsto g.r$ respectively $(g, m) \mapsto g.m$ (such that, for each g in G the maps $r \mapsto g.r$ and $m \mapsto g.m$ are homomorphisms of modules or rings and monoids, respectively). We set

$$R[[M]]^G := \{C : M \rightarrow R : \forall g \in G, m \in M : C(g.m) = g.C(m)\}$$

Note that $R[[M]]^G$, for a ring R , is a subring of the ring of power series $R[[M]]$.

In fact, the types of automorphic forms mentioned above and many more possess a Fourier expansion whose Fourier coefficients lie in $R[[M]]^G$ for suitable choices of M , R and G . Table 4 summarizes various examples. An implementation of these objects would again be a class built around a dictionary whose keys are representatives for the orbits in $G \backslash M$. The effectiveness of such an implementation depends, of course, on a good reduction theory for finding distinguished representatives for the classes in $G \backslash M$. Such a class would be highly desirable and encourage the implementation of more types of automorphic forms. An implementation for Sage by the first author will appear soon.

Most useful for explicit computations are those graded (modules over) rings of automorphic forms of the types mentioned in the table where generators are known and can be computed effectively. There are known examples, in addition to the one of Siegel modular forms of degree 2 already considered, of spaces of Hilbert modular forms, Jacobi forms, and Siegel modular forms of higher degree whose generators are known. More recent examples include Hermitian modular forms on the full hermitian modular group

$$\Gamma_L := \{M \in M(4, \mathbb{Z}_L) : \overline{M}^t J M = J\}$$

Elliptic modular forms	Group G	1
	Monoid M	$\mathbb{Z}_{\geq 0}$
	G -action on M	
	Module (or ring) R	\mathbb{F}
	G -action on R	
Vector-valued Siegel modular forms of degree n and weight k, j	Group G	$\mathrm{GL}(n, \mathbb{Z})$
	Monoid M	Set of semi-positive definite integral quadratic forms f in n variables
	G -action on M	$(g, f) \mapsto f((X_1, \dots, X_n)g)$
	Module (or ring) R	$\mathbb{F}[X_1, \dots, X_n]_j$
	G -action on R	$(g, p) \mapsto \det(g)^k \times p((X_1, \dots, X_n)g)$
Hilbert modular forms of (parallel) weight k over a totally real number field L	Group G	\mathbb{Z}_L^*
	Monoid M	Set of totally positive or zero elements in the inverse different of L
	G -action on M	$(g, a) \mapsto g^2 a$
	Module (or ring) R	\mathbb{F}
	G -action on R	$(g, r) \mapsto N(g)^k r$
Hermitian modular forms over the imaginary quadratic field L	Group G	$\mathrm{GL}(n, \mathbb{Z}_L)$
	Monoid M	Set of semi-positive definite integral hermitian forms f over L with n variables
	G -action on M	$(g, f) \mapsto f((X_1, \dots, X_n)g)$
	Module (or ring) R	\mathbb{F}
	G -action on R	$(g, p) \mapsto \det(g)^k p$
Jacobi forms of weight k and index $m \gg 0$ (in ϑ^{-1} , where ϑ denotes the different of L) over a number field L (cf. [7])	Group G	$\mathbb{Z}_L^* \ltimes 2m\mathbb{Z}_L$
	Monoid M	Set of (D, r) in $\vartheta^{-2} \times \vartheta^{-1}$ s.t. $D \equiv r^2 \pmod{4m\vartheta^{-1}}$, $D = 0$ or $-D \gg 0$
	G -action on M	$((\varepsilon, x), (D, r)) \mapsto ((\varepsilon^2 D, \varepsilon(r+x)))$
	Module (or ring) R	\mathbb{F}
	G -action on R	$((\varepsilon, x), r) \mapsto N(\varepsilon)^k r$

TABLE 4. *Types of Fourier coefficients for various types of automorphic forms (\mathbb{F} denotes a commutative ring)*

with $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and where \mathbb{Z}_L is either the maximal order in $L = \mathbb{Q}(\sqrt{-1})$ [1], [10] or the maximal order in $L = \mathbb{Q}(\sqrt{-3})$ [10]. The generators can, in these cases, be obtained as Hermitian Maass lifts or Hermitian theta series.

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